CA Common Core State Standards

Standards for Mathematical Practice and Transformation Geometry

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Notes

Notes

California Common Core State Standards (CaCCSS)

- A collaborative effort of mathematicians, educators, and scientists from around the nation
- Rigorous, research-based standards for English-language arts and mathematics for grades K-12
- Designed to prepare the nations students with the knowledge and skills needed for success in college and the workforce
- Well-planned and detailed set of content standards; coherent progression of math content through the grades
- California: added an 8^{th}-grade Algebra I course; includes 23 of 28 CCSS 8^{th} grade standards, with additional high school (algebra) standards
- Combination of content standards and standards for mathematical practice

Notes

Implementation Timeline

- Released June 2, 2010
- Adopted by California (with additions) - August 2010
- California joined the SMARTER Balanced Assessment Consortium (SBAC) - May 2011
- Assessments currently being constructed and drafts are circulating. Draft items for some 8^{th} grade topics are taken from MARS and PISA, and some are written by SBAC
- Field testing of assessments - June 2013 and 2014 (not yet known which districts)
- Full implementation of assessments - June 2015

Notes

Background

The Standards for Mathematical Practice are based on

- NCTM Process Standards: problem solving, reasoning and proof, communication, connections, and representation. See `http://www.nctm.org/standards/content.aspx?id=322`
- Strands of Mathematical Proficiency from Adding it Up: conceptual understanding, adaptive reasoning, strategic competence, procedural fluency, and productive disposition. See `http://www.nap.edu/catalog.php?record_id=9822forafreedownloadofthebook`

These can be implemented to strengthen teaching of the current standards, and used to strengthen teaching of current standards while preparing for the Common Core State Standards.

Notes

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them
2. Reason abstractly and quantitatively
3. Construct viable arguments and critique the reasoning of others
4. Model with mathematics
5. Use appropriate tools strategically
6. Attend to precision
7. Look for and make use of structure
8. Look for and express regularity in repeated reasoning

Let's examine one of the Standards for Mathematical Practice.

Notes

On your chart paper...

1. Create a frame with four sections:
   - Essential characteristics
   - Examples of student actions
   - Examples of teaching actions
   - Non-examples
2. Read SMP #2: Reason abstractly and quantitatively.
3. In your groups, brainstorm some items to list in each section.

Notes

2. Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualizeto abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referentsand the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

Notes

Some words of warning...

Food for thought and possible misconceptions:

- **Modeling with mathematics**: this is frequently misunderstood to mean "use manipulatives." Read the standard carefully! It (more or less) means "use mathematics to model the world."
- **Reason abstractly and quantitatively**: What does "reason quantitatively" mean? How is this different from what we do today?
- **Use appropriate tools strategically**: What role does "strategically" play here? What is its importance?

Notes

Transformation Geometry

It had for a long time been evident to me that geometry can in no way be viewed, like arithmetic or combination theory, as a branch of mathematics; instead geometry relates to something already given in nature, namely, space.

Hermann Grassmann, Die Ausdehnungslehre, 1844

Notes

New content and new approach

Here is an example of a dilation with a hands-on activity to help understand the underlying concept.

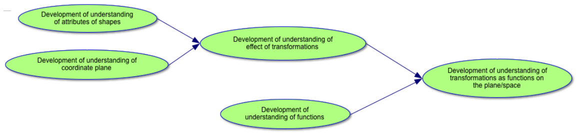
1. Using 2 small rubber bands, link them together. Anchor one end of one band at point P with a pencil. Put another pencil in the other end of the second band. Adjust point P so that you have a good taut band from P to the second pencil.
2. Carefully move the pencil so that the knot linking the bands traces over the figure. You will create a new figure that is (approximately) a dilation of the first.
3. Label the image figure $A'B'C'D'$. Be sure to mark corresponding points with the same letter.

Notes

What does the CaCCSS Geometry domain expect of students?

The CaCCSS expects mathematically proficient geometry students to to *explain*, *visualize*, *understand*, *derive*, and *translate* between representations. They are expected to exhibit geometric habits of mind, and be proficient in the Standards of Mathematical Practice.

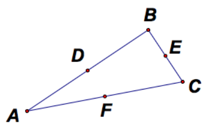
This expanded view of student proficiencies will require that teachers have *greater flexibility and deeper content knowledge*. They will need to employ a greater variety of teaching strategies, and weave deep mathematical content and mathematical habits of mind into a well-orchestrated whole. This is no simple feat! Below is a schematic describing a progression of geometric understanding through the grades:



Notes

Dilations: A Paper and Pencil Example

- On the handout with ΔABC, mark a point P not too far away from the triangle.



- Use your ruler to create rays from P through each of the points A, B, C, D, E, F
- Mark points A', B', C', D', E', F' on the appropriate rays, so that $m(\overline{PA'}) = 2m(\overline{PA})$ and so on.
- Connect points A', B', C' to form a new triangle $\Delta A'B'C'$.

Notes

Some Questions to Consider

- Examine your new triangle $\Delta A'B'C'$. What do you notice about the points D', E', F' in relation to the new triangle $\Delta A'B'C'$?
- Compare the distances $\overline{PA}$ and $\overline{PA'}$. Do the same for $\overline{PB}$ and $\overline{PB'}$, and for $\overline{PC}$ and $\overline{PC'}$. Write a sentence stating your findings, and then compare with your neighbor.
- Examine the sides of the two triangles. Find two characteristics that describe how the various lengths $\overline{AB}$, $\overline{A'B'}$, $\overline{BC}$, $\overline{B'C'}$ $\overline{AC}$, $\overline{A'C'}$ are related.
- What can you say about the relationships between the following quantities?
  - The perimeters of the triangles ΔABC and $\Delta A'B'C'$
  - The areas of the triangles ΔABC and $\Delta A'B'C'$

Notes

Responses to some of the previous questions

- Corresponding angles are congruent.
- Corresponding sides are proportional.
- The lengths of the segments between the new points and O is double the length of the segments between the original points and O.
- The ratio of the perimeter of the image, $\Delta A'B'C'$, to the perimeter of the original, ΔABC, is *the same as* the ratio of corresponding sides.
- The ratio of the area of the image, $\Delta A'B'C'$, to the area of the original, ΔABC, is *the square of* the ratio of corresponding sides

Notes

But what about the rest of the plane?

Notes

- Now imagine that ALL the coordinate points on the plane were shifted in the same way.
  - What would happen to a line segment ST?
  - What would happen to a square $\square WXYZ$ with side length 3?
  - What would happen to a circle with center at O and radius r?
  - What would happen to ANY figure in the plane?

  Discuss these questions with your neighbors and make some conjectures.

Notes

An Important Point!

A geometric transformation transforms *the whole plane* – not just a particular figure.

How does this mathematical concept relate to your current teaching?

Motivation

Notes

What is the problem with the following definition?

Definition

Two figures are similar *if corresponding angles are congruent and corresponding sides are proportional.*

Problem: What can we do if the figures are not rectilinear? Is there no concept of similarity for such figures?

Defining Congruence and Similarity Today

Notes

Definition

1. *A* transformation *of the plane is a one-to-one mapping of the plane onto itself.*
2. *A transformation is a* rigid motion *if it preserves distances between points.*
3. *Two figures are* congruent *if they are related by a rigid motion.*
4. *Two figures are* similar *if they are related by a rigid motion followed by a* dilation *centered in a point.*

Try this! Using these definitions, prove that any two circles are similar and that any two circles of equal radius are congruent.

Food for thought: Show that any two parabolas are similar.

What are Geometric Transformations?

Notes

Dilations are special cases of *geometric transformations* – one-to-one functions from a geometric space onto itself. In the CaCCSS there is significant study of geometric transformations, first *linear transformations* from $\mathbb{R}^2$ onto itself, and later on of $\mathbb{R}^3$ onto itself. Linear transformations have some very useful properties:

Some properties of linear transformations

1. The *identity transformation* is a linear transformation.
2. Every linear transformation f has an *inverse*, and the inverse is a linear transformation.
3. If f and g are linear transformations, then their *composition* $f \circ g$ is a linear transformation.
4. Linear transformations *preserve midpoints*.
5. Linear transformations *transform parallel lines to parallel lines*.

Back to the dilation activity in the coordinate plane.

Notes

Compare this activity to the current geometry standards:

1. What mathematical content appears in this activity that is *different* from today's geometry standards?
2. What mathematical content appears in this activity that is *the same as* today's geometry standards?
3. In what way is the *approach* to doing geometry different in the dilation activity?

How are the definitions below the same? How are they different?

"Static" Definition of Similar Triangles (CA 97)

Two figures in the plane are similar if and only if the ratio between corresponding edges is constant and corresponding angles are congruent.

Definition of Similar Triangles via Transformations (CaCCSS)

Given any two figures in the plane, they are similar if and only if there is a similarity transformation from one onto the other; conversely, every similarity transformation maps a figure onto a similar figure.

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What about congruence?

Is congruence in CaCCSS also viewed differently from CA97?

Notes

Reflection in a Line

A fundamental locus problem:
Given two distinct points A and B, find all points P such that $AP = BP$.

In one dimension, there is only one such point, the midpoint P of $\overline{AB}$.

In two dimensions, this locus is the perpendicular bisector of $\overline{AB}$.

In three dimensions, ... ??

Definition

Let l be a given line in the plane. For any point A in the plane the reflection of A in l *is the point A' such that l is the perpendicular bisector of segment AA'.*

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Composition of Two Reflections

Notes

In the Euclidean plane, two distinct lines either intersect at a unique point or are parallel.

Given two distinct lines, what will happen if we reflect the plane first in one line, and then in the other?

Try it out (for both situations of lines), and see!

Composition of Two Reflections, cont'd

Notes

In the Euclidean plane, two distinct lines either intersect at a unique point or are parallel.

Given two distinct lines, what will happen if we reflect the plane first in one line, and then in the other?

Definition

1. A *translation* is a composition of two reflections of the plane in parallel lines.
2. A *rotation* is a composition of two reflections of the plane in intersecting lines.

Can you reconcile these definitions with those you know intuitively and from previous study?

Released Item from Geometry CST (CA 97 Standards)

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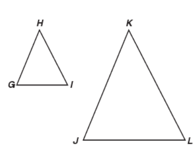
Which of the following statements must be true if $\triangle GHI \sim \triangle JKL$?



A The two triangles must be scalene.

B The two triangles must have exactly one acute angle.

C At least one of the sides of the two triangles must be parallel.

D The corresponding sides of the two triangles must be proportional.

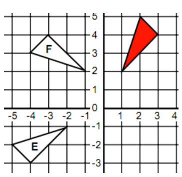
CaCCSS (8^{th} Grade) Geometry Standard

3. Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates.
4. Understand that a two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations; given two similar two-dimensional figures, describe a sequence that exhibits the similarity between them.

Illustration of this standard in
`http://illustrativemathematics.org/standards/k8`

Notes

A sample item in the spirit of CaCCSS, from the MARS Shell Center



1. Draw the shaded triangle after:
   a) It has been translated −7 horizontally and +1 vertically. Label your answer *A*.
   b) It has been reflected over the x-axis. Label your answer *B*.
   c) It has been rotated 90° clockwise around the origin. Label your answer *C*.
   d) It has been reflected over the line $y = x$. Label your answer *D*.
2. Describe fully the single transformation that:
   a) Takes the shaded triangle onto the triangle labeled *E*.
   b) Takes the shaded triangle onto the triangle labeled *F*.
3. Describe a single transformation that has the same effect as rotating a shape 90° clockwise around the origin, then reflecting the result over the x-axis.

Notes

Euclid's Approach to Congruence and Pappus's Interpretation

Euclid's approach (300 BCE):

1. Congruence of plane figures is *not* defined.
2. Instead, propositions are worded in terms of necessary and sufficient conditions for figures to have "equal corresponding parts."
3. What are these "parts"? They are the segments that make up rectilinear figures.

Six hundred years after Euclid, Pappus (300 CE) observed that:

The axiom asserting that all right angles are equal implicitly asserted that...

Congruence of two figures means that *one can be superimposed on the other by a rigid motion*...

The most basic of which he considered to be *reflection in a line*.
This last idea proved foundational to the modern formulation of transformation geometry.

Pappus did not have the modern language of transformations to work with but his proofs were modern in spirit.

Notes

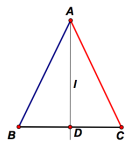
Pappus' proof of the first assertion in Euclid **I.5**

Theorem

The base angles of an isosceles triangle are equal.

Proof. (We give the core idea of the proof here.) Denote

- ΔABC a triangle with $AB = AC$,
- l the bisector of $\angle BAC$, and
- D the intersection of line BC with line l.



Consider the reflection of the plane in line l. How does this lead to a proof?

Notes

What about the converse?

Also Euclid **I.6**, the converse to Euclid **I.5**, was proved by Pappus in the spirit of reflection.

Theorem

If in a triangle two angles are equal to one another, the sides opposite these angles are also equal to one another.

Example: Denote the triangle ΔABC, and suppose that $\angle ABC \cong \angle ACB$. Prove this theorem using reflection in the perpendicular bisector of segment BC.

Notes

Summary Thoughts

1. As a goal, we want students to appreciate that transformations are functions on <u>all</u> the points of the space. More generally, we want students to see that a graph in (for example) $\mathbb{R}^2$ is a collection of special points in the plane. Build up to this understanding with a solid foundation in Euclidean geometry!
2. Have students use tools to build up their spatial intuition. Modern software is a great tool, but we should also encourage the use of standard tools such as straightedge and compass, as well as physical manipulatives.
3. Try to get students to appreciate that transformations are built up from fundamental types, such as reflection for congruence and dilation for similarity.
4. The importance of exploration cannot be overemphasized for the mastery of the above three goals.
5. Above all, especially if transformation geometry is relatively new to you, develop **your** skills and understanding of the fundamental ideas so that you can respond effectively to **students** discoveries and misconceptions.

Notes

Standards for Mathematical Practice and Transformation Geometry: Reflection

- Re-read SMP #2: Reason abstractly and quantitatively. Where did we use this SMP in thinking about transformation geometry today?
- How would implementation of
  - Standards for Mathematical Practice, and/or
  - aspects of transformation geometry

  in the classroom today affect student learning?

Notes

The Standards and other Resources

- More information and the full (Ca)CCSS
  - `http://www.corestandards.org`
  - CA Standards `http://www.scoe.net/castandards/index.html`
- SMARTER Balanced website `http://www.k12.wa.us/smarter/`
- For information and tools for CCSS (frequently updated) `http://commoncoretools.wordpress.com/`
- Website in development with sample tasks `http://illustrativemathematics.org/`
- Tools including sample lesson plans and assessment items `http://map.mathshell.org/materials/`
- Common Core Geometry Taskforce `http://caccssm.cmpso.org/`

Notes

Thank you!

Questions?

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